Workshop Solutions to Section 5.3

1) $\lim_{x\to 0}\frac{x^3+5x^2}{x^2}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 0}\frac{x^3+5x^2}{x^2}=\lim_{x\to 0}\frac{\frac{d}{dx}(x^3+5x^2)}{\frac{d}{dx}(x^2)}=\lim_{x\to 0}\frac{3x^2+10x}{2x}=\frac{0}{0}$$

We obtained an indeterminate form; we can also use the l'Hopital's Rule once more.

$$\lim_{x\to 0}\frac{3x^2+10x}{2x}=\lim_{x\to 0}\frac{\frac{d}{dx}(3x^2+10x)}{\frac{d}{dx}(2x)}=\lim_{x\to 0}\frac{6x+10}{2}$$

$$=\frac{6(0)+10}{2}=\frac{10}{2}=5$$

Another solution:

$$\lim_{x\to 0}\frac{x^3+5x^2}{x^2}=\lim_{x\to 0}\frac{x^2(x+5)}{x^2}=\lim_{x\to 0}(x+5)=0+5=5$$

2) $\lim_{x\to 6}\frac{x-6}{x^2-36}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 6}\frac{x-6}{x^2-36}=\lim_{x\to 6}\frac{\frac{d}{dx}(x-6)}{\frac{d}{dx}(x^2-36)}=\lim_{x\to 6}\frac{1}{2x}=\frac{1}{2(6)}=\frac{1}{12}$$

Another solution:

$$\lim_{x\to 6}\frac{x-6}{x^2-36}=\lim_{x\to 6}\frac{x-6}{(x-6)(x+6)}$$

$$=\lim_{x\to 6}\frac{1}{x+6}=\frac{1}{(6)+6}=\frac{1}{12}$$

3) $\lim_{x\to 1}\frac{x-1}{\ln x}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 1}\frac{x-1}{\ln x}=\lim_{x\to 1}\frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(\ln x)}=\lim_{x\to 1}\frac{1}{\frac{1}{x}}=\lim_{x\to 1}x=1$$

4) $\lim_{x\to\infty}\frac{\ln x}{e^x}=\left(\text{of the form } \frac{\infty}{\infty}\right)\left(\lim_{x\to\infty}\ln x=\infty\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to\infty}\frac{\ln x}{e^x}=\lim_{x\to\infty}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(e^x)}=\lim_{x\to\infty}\frac{\frac{1}{x}}{e^x}$$

$$=\lim_{x\to\infty}\frac{1}{xe^x}=\frac{1}{\infty.\infty}=\frac{1}{\infty}=0$$

5) $\lim_{x\to -6}\frac{x+6}{x^2-36}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to -6}\frac{x+6}{x^2-36}=\lim_{x\to -6}\frac{\frac{d}{dx}(x+6)}{\frac{d}{dx}(x^2-36)}=\lim_{x\to -6}\frac{1}{2x}=\frac{1}{2(-6)}$$

$$=-\frac{1}{12}$$

Another solution:

$$\lim_{x\to -6}\frac{x+6}{x^2-36}=\lim_{x\to -6}\frac{x+6}{(x-6)(x+6)}$$

$$=\lim_{x\to -6}\frac{1}{x-6}=\frac{1}{(-6)-6}=-\frac{1}{12}$$

6) $\lim_{x\to 3}\frac{x^3-27}{x-3}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 3}\frac{x^3-27}{x-3}=\lim_{x\to 3}\frac{\frac{d}{dx}(x^3-27)}{\frac{d}{dx}(x-3)}=\lim_{x\to 3}\frac{3x^2}{1}=\lim_{x\to 3}(3x^2)$$

$$=3(3)^2=27$$

Another solution:

$$\lim_{x\to 3}\frac{x^3-27}{x-3}=\lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{x-3}$$

$$=\lim_{x\to 3}(x^2+3x+9)=(3)^2+3(3)+9=27$$

7) $\lim_{x\to\infty}\frac{x^2}{2e^x} = \left(\text{of the form } \frac{\infty}{\infty}\right)\left(\lim_{x\to\infty} e^x = \infty\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to\infty}\frac{x^2}{2e^x} = \lim_{x\to\infty}\frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(2e^x)} = \lim_{x\to\infty}\frac{2x}{2e^x} = \lim_{x\to\infty}\frac{x}{e^x} = \frac{\infty}{\infty}$$

We obtained an indeterminate form; we can also use the l'Hopital's Rule once more.

$$\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{\frac{d}{dx}(x)}{\frac{d}{dx}(e^x)} = \lim_{x\to\infty}\frac{1}{e^x} = \frac{1}{\infty} = 0$$

8) $\lim_{x\to-2}\frac{x+2}{x^3+8} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to-2}\frac{x+2}{x^3+8} = \lim_{x\to-2}\frac{\frac{d}{dx}(x+2)}{\frac{d}{dx}(x^3+8)} = \lim_{x\to-2}\frac{1}{3x^2} = \frac{1}{3(-2)^2}$$

$$= \frac{1}{12}$$

Another solution:

$$\lim_{x\to-2}\frac{x+2}{x^3+8} = \lim_{x\to-2}\frac{x+2}{(x+2)(x^2-2x+4)}$$

$$= \lim_{x\to-2}\frac{1}{x^2-2x+4} = \frac{1}{(-2)^2-2(-2)+4} = \frac{1}{4+4+4} = \frac{1}{12}$$

9) $\lim_{x\to0^+}\frac{x-\tan x}{x\tan x} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to0^+}\frac{x-\tan x}{x\tan x} = \lim_{x\to0^+}\frac{\frac{d}{dx}(x-\tan x)}{\frac{d}{dx}(x\tan x)}$$

$$= \lim_{x\to0^+}\frac{1-\sec^2 x}{(1)(\tan x)+(x)(\sec^2 x)}$$

$$= \lim_{x\to0^+}\frac{1-\sec^2 x}{\tan x + x\sec^2 x}$$

$$= \frac{1-\sec^2(0)}{\tan(0)+(0)\sec^2(0)} = \frac{0}{0}$$

We obtained an indeterminate form; we can also use the l'Hopital's Rule again.

$$\lim_{x\to0^+}\frac{1-\sec^2 x}{\tan x + x\sec^2 x}$$

$$= \lim_{x\to0^+}\frac{\frac{d}{dx}(1-\sec^2 x)}{\frac{d}{dx}(\tan x + x\sec^2 x)}$$

$$= \lim_{x\to0^+}\frac{-2\sec x\,.\sec x\,.\,\tan x}{\sec^2 x + [(1)(\sec^2 x) + (x)(2\sec x\,.\sec x\,.\,\tan x)]}$$

$$= \lim_{x\to0^+}\frac{-2\sec^2 x\tan x}{\sec^2 x + \sec^2 x + 2\sec^2 x\tan x}$$

$$= \lim_{x\to0^+}\frac{-2\sec^2 x\tan x}{2\sec^2 x + 2\sec^2 x\tan x}$$

$$= \lim_{x\to0^+}\frac{2\sec^2 x(-\tan x)}{2\sec^2 x(1+\tan x)} = \lim_{x\to0^+}\frac{-\tan x}{1+\tan x}$$

$$= \frac{-\tan(0)}{1+\tan(0)} = \frac{0}{1+0} = 0$$

10) $\lim_{x\to1}\frac{\ln x}{\sin(\pi x)} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to1}\frac{\ln x}{\sin(\pi x)} = \lim_{x\to1}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(\sin(\pi x))} = \lim_{x\to1}\frac{\frac{1}{x}}{\pi\cos(\pi x)}$$

$$= \lim_{x\to1}\frac{1}{\pi x\cos(\pi x)} = \frac{1}{\pi(1)\cos(\pi)} = \frac{1}{\pi.(-1)}$$

$$= -\frac{1}{\pi}$$

11) $\lim_{x\to0}\frac{1-\cos x}{x^2} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to0}\frac{1-\cos x}{x^2} = \lim_{x\to0}\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x^2)} = \lim_{x\to0}\frac{\sin x}{2x} = \frac{0}{0}$$

We obtained an indeterminate form; we can also use the l'Hopital's Rule once more.

$$\lim_{x\to0}\frac{\sin x}{2x} = \lim_{x\to0}\frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(2x)} = \lim_{x\to0}\frac{\cos x}{2} = \frac{\cos(0)}{2} = \frac{1}{2}$$

12) $\lim_{x\to 0}\frac{\sin^{-1}x}{\sin x}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 0}\frac{\sin^{-1}x}{\sin x}=\lim_{x\to 0}\frac{\frac{d}{dx}(\sin^{-1}x)}{\frac{d}{dx}(\sin x)}=\lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}}{\cos x}$$
$$=\lim_{x\to 0}\frac{1}{\sqrt{1-x^2}\cos x}=\frac{1}{\sqrt{1-(0)^2}\cos(0)}$$
$$=\frac{1}{(1).(1)}=1$$

13) $\lim_{x\to\infty}\frac{3^x}{6^x}=\left(\text{of the form } \frac{\infty}{\infty}\right)$

$\left(\lim_{x\to\infty}a^x=\infty,\ a>1,\ \lim_{x\to\infty}a^x=0,\ 0<a<1\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to\infty}\frac{3^x}{6^x}=\lim_{x\to\infty}\frac{\frac{d}{dx}(3^x)}{\frac{d}{dx}(6^x)}=\lim_{x\to\infty}\frac{3^x.\ln 3}{6^x.\ln 6}=\frac{\ln 3}{\ln 6}\lim_{x\to\infty}\frac{3^x}{6^x}=\frac{\infty}{\infty}$$

Note that we get back to the same limit. We use the following way

$$\lim_{x\to\infty}\frac{3^x}{6^x}=\lim_{x\to\infty}\left(\frac{3}{6}\right)^x=\lim_{x\to\infty}\left(\frac{1}{2}\right)^x=0$$

14) $\lim_{x\to\infty}\frac{2^x}{3^x}=\left(\text{of the form } \frac{\infty}{\infty}\right)$

$\left(\lim_{x\to\infty}a^x=\infty,\ a>1,\ \lim_{x\to\infty}a^x=0,\ 0<a<1\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to\infty}\frac{2^x}{3^x}=\lim_{x\to\infty}\frac{\frac{d}{dx}(2^x)}{\frac{d}{dx}(3^x)}=\lim_{x\to\infty}\frac{2^x.\ln 2}{3^x.\ln 3}=\frac{\ln 2}{\ln 3}\lim_{x\to\infty}\frac{2^x}{3^x}=\frac{\infty}{\infty}$$

Note that we get back to the same limit. We use the following way

$$\lim_{x\to\infty}\frac{2^x}{3^x}=\lim_{x\to\infty}\left(\frac{2}{3}\right)^x=0$$

15) $\lim_{x\to\infty}\frac{e^x}{x^2}=\left(\text{of the form } \frac{\infty}{\infty}\right)\left(\lim_{x\to\infty}e^x=\infty\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)}=\lim_{x\to\infty}\frac{e^x}{2x}=\frac{\infty}{\infty}$$

We obtained an indeterminate form; we can also use the l'Hopital's Rule once more.

$$\lim_{x\to\infty}\frac{e^x}{2x}=\lim_{x\to\infty}\frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(2x)}=\lim_{x\to\infty}\frac{e^x}{2}=\frac{e^\infty}{2}=\infty$$

16) $\lim_{x\to 4}\frac{x^2-3x-4}{x-4}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 4}\frac{x^2-3x-4}{x-4}=\lim_{x\to 4}\frac{\frac{d}{dx}(x^2-3x-4)}{\frac{d}{dx}(x-4)}=\lim_{x\to 4}\frac{2x-3}{1}$$
$$=\lim_{x\to 4}(2x-3)=2(4)-3=5$$

Another solution:

$$\lim_{x\to 4}\frac{x^2-3x-4}{x-4}=\lim_{x\to 4}\frac{(x-4)(x+1)}{x-4}$$
$$=\lim_{x\to 4}(x+1)=(4)+1=5$$

17) $\lim_{x\to 3}\frac{x^2+4x-21}{x^2-8x+15}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 3}\frac{x^2+4x-21}{x^2-8x+15}=\lim_{x\to 3}\frac{\frac{d}{dx}(x^2+4x-21)}{\frac{d}{dx}(x^2-8x+15)}$$
$$=\lim_{x\to 3}\frac{2x+4}{2x-8}=\frac{2(3)+4}{2(3)-8}=\frac{10}{-2}=-5$$

Another solution:

$$\lim_{x\to 3}\frac{x^2+4x-21}{x^2-8x+15}=\lim_{x\to 3}\frac{(x+7)(x-3)}{(x-3)(x-5)}$$
$$=\lim_{x\to 3}\frac{x+7}{x-5}=\frac{(3)+7}{(3)-5}=\frac{10}{-2}=-5$$

18) $\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}} = \left(\text{of the form } \frac{\infty}{\infty}\right)\left(\lim_{x\to\infty}\ln x = \infty\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(\sqrt[3]{x})} = \lim_{x\to\infty}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}\left(x^{\frac{1}{3}}\right)} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{3}x^{-\frac{2}{3}}}$$

$$= \lim_{x\to\infty}\frac{3}{x.x^{-\frac{2}{3}}} = \lim_{x\to\infty}\frac{3}{x^{\frac{1}{3}}} = \lim_{x\to\infty}\frac{3}{\sqrt[3]{x}} = \frac{3}{\infty} = 0$$

19) $\lim_{x\to 2}\frac{\sqrt[3]{x+6}-2}{x-2} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 2}\frac{\sqrt[3]{x+6}-2}{x-2} = \lim_{x\to 2}\frac{\frac{d}{dx}\left(\sqrt[3]{x+6}-2\right)}{\frac{d}{dx}(x-2)}$$

$$= \lim_{x\to 2}\frac{\frac{d}{dx}\left((x+6)^{\frac{1}{3}}-2\right)}{\frac{d}{dx}(x-2)} = \lim_{x\to 2}\frac{\frac{1}{3}(x+6)^{-\frac{2}{3}}}{1}$$

$$= \lim_{x\to 2}\frac{1}{3(x+6)^{\frac{2}{3}}} = \lim_{x\to 2}\frac{1}{3\sqrt[3]{(x+6)^2}} = \frac{1}{3\sqrt[3]{(2+6)^2}}$$

$$= \frac{1}{3\sqrt[3]{64}} = \frac{1}{3(4)} = \frac{1}{12}$$

20) $\lim_{x\to 0}\frac{\sqrt{x+25}-5}{x} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 0}\frac{\sqrt{x+25}-5}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}\left(\sqrt{x+25}-5\right)}{\frac{d}{dx}(x)}$$

$$= \lim_{x\to 0}\frac{\frac{1}{2\sqrt{x+25}}}{1} = \lim_{x\to 0}\frac{1}{2\sqrt{x+25}}$$

$$= \frac{1}{2\sqrt{(0)+25}} = \frac{1}{2(5)} = \frac{1}{10}$$

21) $\lim_{x\to 0}\frac{1-\cos x}{x^2+x} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 0}\frac{1-\cos x}{x^2+x} = \lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x^2+x)} = \lim_{x\to 0}\frac{\sin x}{2x+1}$$

$$= \frac{0}{2(0)+1} = 0$$

22) $\lim_{x\to 2}\frac{x-2}{2-\sqrt{6-x}} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 2}\frac{x-2}{2-\sqrt{6-x}} = \lim_{x\to 2}\frac{\frac{d}{dx}(x-2)}{\frac{d}{dx}\left(2-\sqrt{6-x}\right)} = \lim_{x\to 2}\frac{1}{-\frac{-1}{2\sqrt{6-x}}}$$

$$= \lim_{x\to 2}\left(2\sqrt{6-x}\right) = 2\sqrt{6-(2)} = 2(2)$$

$$= 4$$

23) $\lim_{x\to 3}\frac{1-\sqrt{x-2}}{2-\sqrt{x+1}} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 3}\frac{1-\sqrt{x-2}}{2-\sqrt{x+1}} = \lim_{x\to 3}\frac{\frac{d}{dx}\left(1-\sqrt{x-2}\right)}{\frac{d}{dx}\left(2-\sqrt{x+1}\right)} = \lim_{x\to 3}\frac{-\frac{1}{2\sqrt{x-2}}}{-\frac{1}{2\sqrt{x+1}}}$$

$$= \lim_{x\to 3}\frac{2\sqrt{x+1}}{2\sqrt{x-2}} = \lim_{x\to 3}\frac{\sqrt{x+1}}{\sqrt{x-2}} = \frac{\sqrt{(3)+1}}{\sqrt{(3)-2}}$$

$$= \frac{2}{1} = 2$$

24) $\lim_{x\to 4}\frac{x^2-6x+8}{x^2+x-20}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 4}\frac{x^2-6x+8}{x^2+x-20}=\lim_{x\to 4}\frac{\frac{d}{dx}(x^2-6x+8)}{\frac{d}{dx}(x^2+x-20)}$$

$$=\lim_{x\to 4}\frac{2x-6}{2x+1}=\frac{2(4)-6}{2(4)+1}=\frac{2}{9}$$

Another solution:

$$\lim_{x\to 4}\frac{x^2-6x+8}{x^2+x-20}=\lim_{x\to 4}\frac{(x-2)(x-4)}{(x-4)(x+5)}$$

$$=\lim_{x\to 4}\frac{x-2}{x+5}=\frac{(4)-2}{(4)+5}=\frac{2}{9}$$

25) $\lim_{x\to -2}\frac{x^3+8}{x^2-x-6}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to -2}\frac{x^3+8}{x^2-x-6}=\lim_{x\to -2}\frac{\frac{d}{dx}(x^3+8)}{\frac{d}{dx}(x^2-x-6)}$$

$$=\lim_{x\to -2}\frac{3x^2}{2x-1}=\frac{3(-2)^2}{2(-2)-1}=\frac{12}{-5}=-\frac{12}{5}$$

Another solution:

$$\lim_{x\to -2}\frac{x^3+8}{x^2-x-6}=\lim_{x\to -2}\frac{(x+2)(x^2-2x+4)}{(x-3)(x+2)}$$

$$=\lim_{x\to -2}\frac{x^2-2x+4}{x-3}=\frac{(-2)^2-2(-2)+4}{(-2)-3}=\frac{4+4+4}{-5}=-\frac{12}{5}$$

26) $\lim_{x\to -2}\frac{4x^2+6x-4}{2x^2-8}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to -2}\frac{4x^2+6x-4}{2x^2-8}=\lim_{x\to -2}\frac{\frac{d}{dx}(4x^2+6x-4)}{\frac{d}{dx}(2x^2-8)}$$

$$=\lim_{x\to -2}\frac{8x+6}{4x}=\frac{8(-2)+6}{4(-2)}=\frac{-10}{-8}=\frac{5}{4}$$

Another solution:

$$\lim_{x\to -2}\frac{4x^2+6x-4}{2x^2-8}=\lim_{x\to -2}\frac{2(2x^2+3x-2)}{2(x^2-4)}$$

$$=\lim_{x\to -2}\frac{2x^2+3x-2}{x^2-4}=\lim_{x\to -2}\frac{(2x-1)(x+2)}{(x-2)(x+2)}=\lim_{x\to -2}\frac{2x-1}{x-2}$$

$$=\frac{2(-2)-1}{(-2)-2}=\frac{-5}{-4}=\frac{5}{4}$$

27) $\lim_{x\to 1}\frac{\sqrt{2x+2}-2}{\sqrt{3x-2}-1}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 1}\frac{\sqrt{2x+2}-2}{\sqrt{3x-2}-1}=\lim_{x\to 1}\frac{\frac{d}{dx}\left(\sqrt{2x+2}-2\right)}{\frac{d}{dx}\left(\sqrt{3x-2}-1\right)}$$

$$=\lim_{x\to 1}\frac{\frac{2}{2\sqrt{2x+2}}}{\frac{3}{2\sqrt{3x-2}}}=\lim_{x\to 1}\frac{2\sqrt{3x-2}}{3\sqrt{2x+2}}$$

$$=\frac{2\sqrt{3(1)-2}}{3\sqrt{2(1)+2}}=\frac{2(1)}{3(2)}=\frac{2}{6}=\frac{1}{3}$$

28) $\lim\limits_{x\to -1}\dfrac{x^2-5x-6}{x+1}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to -1}\frac{x^2-5x-6}{x+1}=\lim_{x\to -1}\frac{\frac{d}{dx}(x^2-5x-6)}{\frac{d}{dx}(x+1)}$$
$$=\lim_{x\to -1}\frac{2x-5}{1}=2(-1)-5=-7$$

29) $\lim\limits_{x\to 0}\dfrac{(x+3)^{-1}-3^{-1}}{x}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 0}\frac{(x+3)^{-1}-3^{-1}}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}[(x+3)^{-1}-3^{-1}]}{\frac{d}{dx}(x)}$$
$$=\lim_{x\to 0}\frac{-(x+3)^{-2}}{1}=\lim_{x\to 0}\frac{-1}{(x+3)^2}$$
$$=-\frac{1}{(0+3)^2}=-\frac{1}{9}=-9^{-1}$$

30) $\lim\limits_{x\to\infty}\dfrac{4x^5+6x-4}{2x^5-8}=\left(\text{of the form } \frac{\infty}{\infty}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to\infty}\frac{4x^5+6x-4}{2x^5-8}=\lim_{x\to\infty}\frac{\frac{d}{dx}(4x^5+6x-4)}{\frac{d}{dx}(2x^5-8)}$$
$$=\lim_{x\to\infty}\frac{20x^4+6}{10x^4}=\lim_{x\to\infty}\left(\frac{20x^4}{10x^4}+\frac{6}{10x^4}\right)$$
$$=\lim_{x\to\infty}\left(2+\frac{6}{10x^4}\right)=2+\frac{6}{\infty}=2+0=2$$

31) $\lim\limits_{x\to\infty}\dfrac{4x^4+6x-4}{2x^5-8}=\left(\text{of the form } \frac{\infty}{\infty}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to\infty}\frac{4x^4+6x-4}{2x^5-8}=\lim_{x\to\infty}\frac{\frac{d}{dx}(4x^4+6x-4)}{\frac{d}{dx}(2x^5-8)}$$
$$=\lim_{x\to\infty}\frac{16x^3+6}{10x^4}=\lim_{x\to\infty}\left(\frac{16x^3}{10x^4}+\frac{6}{10x^4}\right)$$
$$=\lim_{x\to\infty}\left(\frac{8}{5x}+\frac{6}{10x^4}\right)=\frac{8}{\infty}+\frac{6}{\infty}=0+0=0$$

32) $\lim\limits_{x\to 0^+}\dfrac{\ln x}{x^{-1}}=\left(\text{of the form } \frac{-\infty}{\infty}\right)\left(\lim\limits_{x\to 0^+}\ln x=-\infty\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 0^+}\frac{\ln x}{x^{-1}}=\lim_{x\to 0^+}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{-1})}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-x^{-2}}$$
$$=\lim_{x\to 0^+}\frac{x^2}{-x}=\lim_{x\to 0^+}(-x)=0$$

33) $\lim\limits_{x\to 0^+}\dfrac{\ln(x+1)}{x}=\left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 0^+}\frac{\ln(x+1)}{x}=\lim_{x\to 0^+}\frac{\frac{d}{dx}(\ln(x+1))}{\frac{d}{dx}(x)}=\lim_{x\to 0^+}\frac{\frac{1}{x+1}}{1}$$
$$=\lim_{x\to 0^+}\frac{1}{x+1}=\frac{1}{(0)+1}=1$$

34) $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}=\left(\text{of the form } \frac{\infty}{\infty}\right)\left(\lim\limits_{x\to\infty}\ln x=\infty\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}$$
$$=\lim_{x\to\infty}\frac{1}{x}=\frac{1}{\infty}=0$$

35) $\lim_{x\to 1^+} \frac{1-x+x\ln x}{(x-1)\ln x} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 1^+} \frac{1-x+x\ln x}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{\frac{d}{dx}(1-x+x\ln x)}{\frac{d}{dx}((x-1)\ln x)}$$

$$= \lim_{x\to 1^+} \frac{-1+\left(\ln x + x.\frac{1}{x}\right)}{\ln x + (x-1).\frac{1}{x}} = \lim_{x\to 1^+} \frac{-1+\ln x+1}{\ln x + 1 - \frac{1}{x}}$$

$$= \lim_{x\to 1^+} \frac{\ln x}{\frac{x\ln x + x - 1}{x}} = \lim_{x\to 1^+} \frac{x\ln x}{x\ln x + x - 1} = \frac{0}{0}$$

We obtained an indeterminate form; we can also use the l'Hopital's Rule again.

$$\lim_{x\to 1^+} \frac{x\ln x}{x\ln x + x - 1} = \lim_{x\to 1^+} \frac{\frac{d}{dx}(x\ln x)}{\frac{d}{dx}(x\ln x + x - 1)}$$

$$= \lim_{x\to 1^+} \frac{\ln x + x.\frac{1}{x}}{\left(\ln x + x.\frac{1}{x}\right) + 1}$$

$$= \lim_{x\to 1^+} \frac{\ln x + 1}{(\ln x + 1) + 1}$$

$$= \lim_{x\to 1^+} \frac{\ln x + 1}{\ln x + 2} = \frac{\ln(1)+1}{\ln(1)+2} = \frac{0+1}{0+2} = \frac{1}{2}$$

36) $\lim_{x\to\infty} \frac{3^x}{2^x} = \left(\text{of the form } \frac{\infty}{\infty}\right)$

$\left(\lim_{x\to\infty} a^x = \infty,\ a > 1,\ \lim_{x\to\infty} a^x = 0,\ 0 < a < 1\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to\infty} \frac{3^x}{2^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(3^x)}{\frac{d}{dx}(2^x)} = \lim_{x\to\infty} \frac{3^x.\ln 3}{2^x.\ln 2} = \frac{\ln 3}{\ln 2} \lim_{x\to\infty} \frac{3^x}{2^x} = \frac{\infty}{\infty}$$

Note that we get back to the same limit. We use the following way

$$\lim_{x\to\infty} \frac{3^x}{2^x} = \lim_{x\to\infty} \left(\frac{3}{2}\right)^x = \infty$$

37) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^3} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^3} = \lim_{x\to 0} \frac{\frac{d}{dx}(e^x - 1 - x)}{\frac{d}{dx}(x^3)}$$

$$= \lim_{x\to 0} \frac{e^x - 1}{3x^2} = \frac{0}{0}$$

We obtained an indeterminate form; we can also use the l'Hopital's Rule again.

$$\lim_{x\to 0} \frac{e^x - 1}{3x^2} = \lim_{x\to 0} \frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}(3x^2)} = \lim_{x\to 0} \frac{e^x}{6x} = \frac{e^0}{6(0)} = \frac{1}{0} = \infty$$

38) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 0} \frac{\tan^{-1} x}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(\tan^{-1} x)}{\frac{d}{dx}(x)} = \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1}$$

$$= \lim_{x\to 0} \frac{1}{1+x^2} = \frac{1}{1+(0)^2} = 1$$

39) $\lim_{x\to 0^+} \frac{\sqrt{x} - x}{x\sqrt{x}} = \left(\text{of the form } \frac{0}{0}\right)$

Solution:

We use the l'Hopital's Rule, we have

$$\lim_{x\to 0^+} \frac{\sqrt{x} - x}{x\sqrt{x}} = \lim_{x\to 0^+} \frac{\frac{d}{dx}(\sqrt{x} - x)}{\frac{d}{dx}(x\sqrt{x})} = \lim_{x\to 0^+} \frac{\frac{1}{2\sqrt{x}} - 1}{\sqrt{x} + x.\frac{1}{2\sqrt{x}}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1-2\sqrt{x}}{2\sqrt{x}}}{\frac{2x+x}{2\sqrt{x}}} = \lim_{x\to 0^+} \frac{1-2\sqrt{x}}{3x}$$

$$= \frac{1-2\sqrt{(0)}}{3(0)} = \frac{1}{0} = \infty$$